

Sharp Weight Hardy-Sobolev inequalities for Grand Lebesgue Spaces.

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Abstract.

We give in this short paper the exact value for norms of two operators of Hardy-Sobolev type acting between two weight Grand Lebesgue Space (GLS) based on the whole multidimensional Euclidean space.

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1. INTRODUCTION. STATEMENT OF PROBLEM.

Let $(X, |x|)$ be ordinary Euclidean space $X = R^n$ with norm $|x| = \sqrt{(x, x)}$. The standard Lebesgue-Riesz $L(p)$, $p \geq 1$ norm of a measurable function $f : X \rightarrow R$ will be denoted

$$|f|_p \stackrel{\text{def}}{=} \left[\int_X |f(x)|^p dx \right]^{1/p}.$$

We recall for beginning some famous and interest inequalities.

A. Hardy-Rellich inequality.

Introduce an (linear) operator (Hardy-Rellich operator)

$$V[f](x) := \frac{f(x)}{|x|^2},$$

then the Hardy-Rellich (-Sobolev) inequality may be written as follows, see [1]

$$|V[f]|_p \leq K_{HR}(n, p) |\Delta f|_p, \tag{1.1}$$

where Δ is the Laplacian and

$$K_{HR}(n, p) := \frac{pp'}{n(n-2p)}, \quad p' := p/(p-1), \tag{1.1a}$$

and $n \geq 3$, $1 < p < n/2$; wherein this constant $K_{HR}(n, p)$ is the best possible:

$$\sup_{f: \Delta f \neq 0} \frac{|V[f]|_p}{|\Delta f|_p} = K_{HR}(n, p). \quad (1.2)$$

Obviously, when $p \rightarrow 1 + 0$

$$K_{HR}(n, p) \sim \frac{(p-1)^{-1}}{n(n-2)},$$

and if $p \rightarrow n/2 - 0$

$$K_{HR}(n, p) \sim \frac{n}{4(n-2)} \cdot \frac{1}{n/2 - p}.$$

Recall that here $n \geq 3$.

B. Weight Sobolev's inequality.

Introduce an (linear) operator

$$W[f](x) = |x|^{-\beta} f(x), \quad x \in R^n, \quad \beta = \text{const} \in (0, n), \quad n \geq 2, \quad p \in (1, n/\beta);$$

then [1]

$$|W[f](x)|_p \leq K_S(n, \beta, p) \cdot |(-\Delta)^{\beta/2} f|_p, \quad (1.3)$$

where the fractional degree of the anti-Laplace operator $(-\Delta)^{\beta/2}$ is defined through the Fourier transform and the best possible "constant" $K_S(n, \beta, p)$ has a form

$$K_S(n, \beta, p) = 2^{-\beta} \frac{\Gamma[(\beta/(2p)) \cdot (n/\beta - p)] \cdot \Gamma[n/2 - n/(2p)]}{\Gamma[(n + \beta)/2 - n/(2p)] \Gamma[n/(2p)]}. \quad (1.3a)$$

Note that as $p \rightarrow 1 + 0$

$$K_S(n, \beta, p) \sim 2^{1-\beta} \frac{\Gamma[(n - \beta)/2] n^{-1}}{\Gamma(\beta/2) \Gamma(n/2)} \cdot (p - 1)^{-1},$$

and as $p \rightarrow n/\beta - 0$

$$K_S(n, \beta, p) \sim 2^{-\beta} \cdot \frac{\beta^2}{2n(n/\beta - p)} \cdot \frac{\Gamma[(n - \beta)/2]}{\Gamma(\beta/2) \Gamma(n/2)}.$$

Our aim is a generalization of the estimation (1.1) and (1.3) on the so-called Grand Lebesgue Spaces $GLS = GLS(\psi) = G(\psi)$, i.e. when $f(\cdot) \in G(\psi)$ and to show the accuracy of obtained estimations.

□

2. BRIEFLY ABOUT GRAND LEBESGUE SPACES.

We recall briefly the definition and needed properties of these spaces. More details see in the works [5], [6], [8], [9], [21], [22], [13], [11], [12] etc. More about rearrangement invariant spaces see in the monographs [3], [14].

For a and b constants, $1 \leq a < b \leq \infty$, let $\psi = \psi(p)$, $p \in (a, b)$, be a continuous positive function such that there exists a limits (finite or not) $\psi(a+0)$ and $\psi(b-0)$, with conditions $\inf_{p \in (a,b)} \psi(p) > 0$ and $\min\{\psi(a+0), \psi(b-0)\} > 0$. We will denote the set of all these functions as $\Psi(a, b)$.

The Grand Lebesgue Space (in notation GLS) $G(\psi; a, b) = G(\psi)$ is the space of all measurable functions $f : R^d \rightarrow R$ endowed with the norm

$$\|f\|_{G(\psi)} \stackrel{def}{=} \sup_{p \in (a,b)} \left[\frac{|f|_p}{\psi(p)} \right], \quad (2.1)$$

if it is finite.

In the article [22] there are many examples of these spaces. For instance, in the case when $1 \leq a < b < \infty, \beta, \gamma \geq 0$ and

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p-a)^{-\beta} (b-p)^{-\gamma};$$

we will denote the correspondent $G(\psi)$ space by $G(a, b; \beta, \gamma)$; it is not trivial, non-reflexive, non-separable etc. In the case $b = \infty$ we need to take $\gamma < 0$ and define

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = (p-a)^{-\beta}, p \in (a, h);$$

$$\psi(p) = \psi(a, b; \beta, \gamma; p) = p^{-\gamma} = p^{-|\gamma|}, p \geq h,$$

where the value h is the unique solution of a continuity equation

$$(h-a)^{-\beta} = h^{-\gamma}$$

in the set $h \in (a, \infty)$.

The $G(\psi)$ spaces over some measurable space (X, F, μ) with condition $\mu(X) = 1$ (probabilistic case) appeared in [13].

The GLS spaces are rearrangement invariant spaces and moreover interpolation spaces between the spaces $L_1(R^d)$ and $L_\infty(R^d)$ under real interpolation method [4], [11].

It was proved also that in this case each $G(\psi)$ space coincides with the so-called *exponential Orlicz space*, up to norm equivalence. In others quoted publications were investigated, for instance, their associate spaces, fundamental functions $\phi(G(\psi; a, b); \delta)$, Fourier and *singular* integral operators acting in these spaces, conditions for convergence and compactness, reflexivity and separability, martingales in these spaces etc.

Remark 2.1. If we introduce the *discontinuous* function

$$\psi_r(p) = 1, p = r; \quad \psi_r(p) = \infty, p \neq r, p, r \in (a, b)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = |f|_r.$$

Thus, the Grand Lebesgue Spaces are the direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \rightarrow R$ such that $\exists(a, b) : 1 \leq a < b \leq \infty, \forall p \in (a, b) |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p. \quad (2.2)$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T$, T is arbitrary set, be some *family* $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \xi(t, \cdot) : X \rightarrow R$ such that

$$\exists(a, b) : 1 \leq a < b \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p. \quad (2.2a)$$

The function $\psi_F(p)$ may be called as a *natural function* for the family F . This method was used in the probability theory, more exactly, in the theory of random fields, see [21].

The GLS norm estimates, in particular, Orlicz norm estimates for measurable functions, e.g., for random variables are used in the theory of Partial Differential Equations [5], [8], theory of probability in Banach spaces [13], [21], in the modern non-parametrical statistics, for example, in the so-called regression problem [21].

We use the symbols $C(X, Y)$, $C(p, q; \psi)$, etc., to denote positive constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like $C_1(X, Y)$ and $C_2(X, Y)$. The relation $g(\cdot) \asymp h(\cdot)$, $p \in (A, B)$, where $g = g(p)$, $h = h(p)$, $g, h : (A, B) \rightarrow R_+$, denotes as usually

$$0 < \inf_{p \in (A, B)} h(p)/g(p) \leq \sup_{p \in (A, B)} h(p)/g(p) < \infty.$$

The symbol \sim will denote usual equivalence in the limit sense. □

3. MAIN RESULT: NORM ESTIMATIONS FOR CONSIDERED OPERATORS.

A. Hardy-Rellich case.

Suppose the (measurable) function $\Delta f(\cdot) : R^n \rightarrow R$ belongs to some space $G\psi(a, b)$, i.e. $\|\Delta f\|_{G\psi} < \infty$, where $\psi(\cdot) \in \Psi(a, b)$, $1 \leq a < b \leq \infty$. For instance, the function $\psi(\cdot)$ may be picked as a natural function for $\Delta f(\cdot)$, if there exists: $\psi(p) = \psi_0(p)$, where

$$\psi_0(p) := \|\Delta f\|_p, \quad p \in (a, b).$$

The set of all such a functions $f : f \in W^2 G\psi = W^2 G\psi(R^n)$ equipped with the semi-norm

$$\| f \|_{W^2 G\psi} \stackrel{def}{=} \| \Delta f \|_{G\psi}$$

is named as usually Sobolev-Grand Lebesgue Space; it is a complete Banach space.

Define the segment

$$I_{HR} = (p_0, p_1) \stackrel{def}{=} (1, n/2) \cap (a, b) \quad (3.1)$$

and suppose its non-triviality: $I_{HR} \neq \emptyset$ or equally $1 < p_0 < p_1 < n/2$.

The most interesting and important case appears, by our opinion, when $a = 1$ and $b = n/2$; then

$$I_{HR} = (1, n/2). \quad (3.2)$$

Let us define a new Ψ – function

$$\psi_V(p) \stackrel{def}{=} K_{HR}(p) \cdot \psi(p), \quad p \in I_{HR}. \quad (3.3)$$

Theorem 3.A.

$$\| V[f] \|_{G\psi_V} \leq 1 \cdot \| f \|_{W^2 G\psi}, \quad (3.4)$$

where the constant "1" in (3.4) is the best possible in the case (3.2).

Proof. Let $f(\cdot) \in W^2 G\psi$; we can and will suppose without loss of generality

$$\| f \|_{W^2 G\psi} = 1.$$

It follows immediately from the direct of the Grand Lebesgue Spaces norm

$$| \Delta f |_p \leq \psi(p), \quad p \in (a, b).$$

We use the inequality (1.1):

$$| V[f] |_p \leq K_{HR}(n, p) | \Delta f |_p \leq K_{HR}(n, p) \cdot \psi(p) =$$

$$\psi_V(p) = \psi_V(p) \cdot \| f \|_{W^2 G\psi},$$

or equally

$$\| V[f] \|_{G\psi_V} \leq \| f \|_{W^2 G\psi}.$$

The sharpness of the constant "1" follows immediately from one of the results of the preprint [23].

B. Sobolev's case.

Suppose now that the (measurable) function $f(\cdot) : R^n \rightarrow R$ is such that $(-\Delta)^{\beta/2} f$ belongs to some space $G\psi(a, b)$, i.e. $\| (-\Delta)^{\beta/2} f \|_{G\psi} < \infty$, where $\psi(\cdot) \in \Psi(a, b)$, $1 \leq a < b \leq \infty$. For instance, the function $\psi(\cdot)$ may be picked as a natural function for $(-\Delta)^{\beta/2} f(\cdot)$, if there exists: $\psi(p) = \psi_1(p)$, where

$$\psi_1(p) := | (-\Delta)^{\beta/2} f |_p, \quad p \in (a, b).$$

The set of all such a functions $f : f \in W^{(\beta)}G\psi = W^{(\beta)}G\psi(R^n)$ equipped with the semi - norm

$$\| f \|_{W^{(\beta)}G\psi} \stackrel{def}{=} \| (-\Delta f)^{\beta/2} \|_{G\psi}$$

is named again as usually Sobolev-Grand Lebesgue Space; it is also the complete Banach space.

Define the segment

$$J_W = (p_2, p_3) \stackrel{def}{=} (1, n/\beta) \cap (a, b) \quad (3.5)$$

and suppose its non-triviality: $J_W \neq \emptyset$ or equally $1 < p_2 < p_3 < n/\beta$.

The most interesting and important case appears, by our opinion, when $a = 1$ and $b = n/\beta$; then

$$J_{HR} = (1, n/\beta). \quad (3.6)$$

Let us define a new Ψ - function

$$\psi_W(p) \stackrel{def}{=} K_W(p) \cdot \psi(p), \quad p \in J_{HR}. \quad (3.7)$$

Analogously to the theorem 3.A may be proved the following result.

Theorem 3.B.

$$\| W[f] \|_{G\psi_W} \leq 1 \cdot \| f \|_{W^{(\beta)}G\psi}, \quad (3.8)$$

where the constant "1" in (3.8) is the best possible in the case (3.6).

□

4. SIMPLIFICATIONS.

Notice that we know

$$K_{HR}(p) \asymp \frac{1}{(p-1)(n/2-p)}, \quad p \in (1, n/2).$$

More precisely,

$$\frac{C_1(n)}{(p-1)(n/2-p)} \leq K_{HR}(p) \leq \frac{C_2(n)}{(p-1)(n/2-p)}, \quad p \in (1, n/2),$$

where as before $n \geq 3$ and

$$0 < \inf_{n \geq 3} C_1(n) < \sup_{n \geq 3} C_2(n) < \infty.$$

Analogously

$$\frac{C_3(n, \beta)}{(p-1)(n/\beta-p)} \leq K_S(p) \leq \frac{C_4(n, \beta)}{(p-1)(n/\beta-p)}, \quad p \in (1, n/\beta),$$

where $n \geq 2$ and

$$0 < \inf_{n \geq 2} C_3(n, \beta) < \sup_{n \geq 2} C_4(n, \beta) < \infty.$$

Therefore, one can in the relations (3.3) and (3.7) the coefficients $K_{HR}(p)$, $K_S(p)$ replace to the more simple expressions. The assertions of theorem (3.A) and (3.B) remain true up to finite multiplicative constants. □

5. CONCLUDING REMARKS

Of course, described here method may be applied to the more wide class of operators, non necessary to be linear, differential or integral, for which are known $L(p) - L(q)$ estimates. Many examples of these operators one can find in articles and books [2], [7], [12], [16], [17], [18], [20], [23]-[24], [25], [26], [27] etc.

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